

## A TORELLI-LIKE THEOREM FOR SMOOTH PLANE CURVES

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ABSTRACT. The Information-Theoretic Schottky Problem treats the period matrix of a compact Riemann Surface as a compressible signal. In this case, the period matrix of a smooth plane curve is characterized by only 4 of its columns, a significant compression.

## 1. INTRODUCTION

Begin by fixing notation; consult [5] as a general reference.

Let  $X$  be a compact Riemann Surface of genus  $g > 1$ ; equivalently,  $X$  is a non-singular complex algebraic curve. Choose a basis  $\omega_1, \dots, \omega_g$  for the space  $H^{1,0}(X)$  of holomorphic differentials on  $X$ , and a symplectic basis  $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$  for the singular homology  $H_1(X, \mathbf{Z})$ , normalized so  $\int_{\alpha_i} \omega_j = \delta_{ij}$ , the Dirac delta. The matrix  $\Omega_{ij} := \int_{\beta_i} \omega_j$  is the *period matrix*; Riemann proved that it is symmetric with positive definite imaginary part. The torus  $\mathbb{C}^g/[I|\Omega]$  is the *Jacobian* of  $X$ . Torelli's Theorem asserts that the Jacobian determines all of the properties of  $X$ . In practice deciding which properties apply is seldom successful (but see [7]).

The period matrix is symmetric with positive-definite imaginary part, and the space of such matrices forms the *Siegel upper half-space*  $H_g$ . Its dimension is  $g(g+1)/2$ , while the dimension of the moduli space of curves of degree  $g$  has dimension  $3g-3$ . Distinguishing the period matrices from arbitrary elements of  $H_g$  is the *Schottky Problem*. See [3] for details on the problem and some of its previous solutions.

Now, recast the problem in terms of communication. Suppose that Alice wants to tell Bob about a curve. By Torelli's Theorem, she can do so by telling him the period matrix, but this means transmitting  $O(g^2)$  complex numbers in order to describe something that depends on  $O(g)$  parameters. In other words, the period matrix is *sparse* in the sense of [2], and should therefore be compressible.

The perspective that the period matrix is a compressible signal is the central idea of the *Information-Theoretic Schottky Problem*. The attempt to apply ideas from Compressed Sensing [2] to the Schottky problem has led to many interesting experiments, conjectures, and theorems [8].

The result described here is purely mathematical, rather than computational; however, it was inspired by an attempt to implement ideas in blind Compressed Sensing, as described in [4].

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## 2. PLANE CURVES

Shift the focus to a smooth plane curve whose affine equation  $f(x, y) = 0$  has degree  $d > 4$ . Its genus is  $g = \frac{(d-1)(d-2)}{2}$ , and its holomorphic differentials are given by  $h(x, y) \frac{dx}{\partial f / \partial y}$ , where  $h$  is a so-called adjoint polynomial of degree  $d - 3$ . Fix an order for the monomials of degree  $d - 3$ , eg the usual lexicographic order  $x^0 y^0 < x^1 y^0 < x^0 y^1 < \dots < x^0 y^{d-3}$ , thus forming a proxy basis for  $H^{(1,0)}(X)$ .

The main result is

**Theorem 2.1.** *There is a set of 4 columns of the period matrix of a smooth plane curve that characterize the curve; in other words, if  $X'$  is another plane curve whose period matrix includes these four columns, then  $X$  and  $X'$  are holomorphically equivalent.*

The four columns involved have  $4g$  entries, so constitute a rather small superset of “moduli.” Thus, this is a significant loss-less compression of the period matrix.

The number 4 seems rather arbitrary, but the condition that a curve have a smooth planar representation is strong; one would not expect such a strong result from weaker hypotheses.

## 3. PERIOD MATRICES AND MODULI

The primary tool relating period matrices to moduli is the following theorem of Rauch. Let  $K$  denote the canonical divisor on  $X$ .

**Theorem 3.1.** *[Rauch] Let  $\{\zeta_1, \dots, \zeta_g\}$  be a normalized basis for  $H^{(1,0)}(X)$  of a non-hyperelliptic Riemann surface  $X$ , and suppose that  $\{\zeta_i \zeta_j : (i, j) \in (I, J)\}$  form a basis for the quadratic differentials  $H^0(X, 2K)$ . If another Riemann surface  $X'$  has the same entries as  $X$  in the  $(I, J)$  positions of its period matrix then  $X$  and  $X'$  are holomorphically equivalent.*

The proof, while not strictly relevant here, may be of interest for further ITSP investigations. It chooses the minimal member of the homotopy class of maps from the underlying surface of  $X$  to the underlying surface of  $X'$  with respect to the Douglas–Dirichlet energy, and proceeds by a delicate argument using infinitesimal quadratic differentials to show that this map is holomorphic.

In principle, then, Alice can send Bob  $3g - 3$  entries of the period matrix, and he can then verify that his period matrix is the same. However, there is no canonical way to choose which  $3g - 3$  elements to send, and there are many choices of  $3g - 3$  elements that do not form moduli. The point of Theorem A, then, is that in the case of a smooth plane curve Alice can canonically choose a slightly larger set of periods to send.

Returning to plane curves, the strategy is to choose a basis for  $H^0(2K)$  carefully; in the end, this will involve only 4 columns of the period matrix.

## 4. PROOF OF THE THEOREM

Recall the theorem of Noether (quoted in [6]; also see [1]) that every quadratic differential is a product of ordinary differentials.

To determine  $(I, J)$ , define a  $g \times g$  matrix  $Q$  whose rows and columns are indexed by the adjoint monomials. In writing the matrix the factor  $dx / \frac{\partial f}{\partial y}$  is omitted,

and in considering quadratic differentials one only need to look at products of the monomials. In the case of degree  $d = 6$ , the curve has genus  $g = 10$  and, filling only the top row and leftmost column,

$$Q = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \\ x & & & & & & & & & \\ y & & & & & & & & & \\ x^2 & & & & & & & & & \\ xy & & & \dots & & & & & & \\ y^2 & & & & & & & & & \\ x^3 & & & & & & & & & \\ x^2y & & & & & & & & & \\ xy^2 & & & & & & & & & \\ y^3 & & & & & & & & & \end{bmatrix}$$

In this case, the columns beginning with  $1$ ,  $x^3$ , and  $y^3$  must be included to form a basis for  $H^0(X, 2K)$ . More generally, the  $x^{d-3}$  and  $y^{d-3}$  columns must be included in order to get all of the monomials with  $x$ -degree (resp.  $y$ -degree) greater than  $d - 3$ . Note that these three columns contain duplicate entries, for example,  $x^{d-3}y^{d-3}$  appears in both the  $x^{d-3}$ - and  $y^{d-3}$ -columns.

The entries in these three columns do not constitute a basis, since they omit the monomials of degree greater than  $d - 3$  but of  $x$ - or  $y$ -degree  $\leq d - 4$ , but all of these monomials are in the  $x^2y^2$  column. To see this, let  $x^r y^{d-2-r}$  be a monomial of degree  $d - 2$ ; here  $r \leq d - 4$ . This monomial factors as  $x^2y^2 \cdot x^{r-2}y^{d-4-r}$ , and  $x^{r-2}y^{d-4-r}$  is a monomial in the first column. Thus,  $x^r y^{d-2-r}$  appears in the  $x^2y^2$  column; the same applies to  $x^{d-2-r}y^r$ .

Similarly, monomials of degree  $d - 1$  can be written  $x^r y^{d-1-r}$ , which factors as  $x^2y^2 \cdot x^{r-2}y^{d-3-r}$ . Clearly  $d - 3 - r \leq d - 3$ , so again such a monomial is a product of  $x^2y^2$  and a monomial from the first column.

The largest-degree monomial satisfying the conditions is  $x^{d-4}y^{d-4} = x^2y^2 \cdot x^{d-6}y^{d-6}$ .

Thus, every “missing” monomial appears in the  $x^2y^2$  column.

Now consider the corresponding entries in the period matrix. Since the differentials of the chosen basis are not normalized, multiply the right half of the period matrix by the inverse of the left half. Each entry from  $(I, J)$  in the normalized period matrix is a thus linear combination of entries from the corresponding column in the matrix associated with  $Q$ . But these entries still correspond to a (superset) of a basis for  $H^0(2K)$ , and thus by the Rauch Theorem determine the curve up to isomorphism.

## 5. COMPLEMENTS

- The four columns contain no more than  $4g$  entries, which is a substantial compression of the period matrix.

Even removing the duplicates, there are other relations among the quadratic differentials and thus more relations between the periods. This is easiest to see in degree 6, where removing duplicate entries leaves 28 positions, while the number of moduli is 27. The missing relation occurs in degree 6, and is, in fact, the equation of the curve. In other words, some of the redundancy from the superset of periods used to determine the curve come from the equation itself.

In higher degrees, many of the redundancies are in the ideal generated by the equation.

- Since the columns of the normalized period matrix each correspond to integrals over a cycle, it appears that only four of the generators of the first singular homology group determine the whole topology of the curve, but this is not the case because of the symmetry of the normalized period matrix.

- It is neither true nor claimed that every set of four columns determines the curve; using the Alice–Bob scenario, Alice, knowing that she has a plane curve, chooses the columns used in the proof and sends them to Bob. Bob also has a curve, or perhaps a period matrix, but may or may not know *a priori* whether his period matrix is a plane curve, but if it contains the four columns then he has determined that the curve that Alice sent is the one he has. In this sense the theorem provides more of a verification than an actual communication.

- D. Litt points out that it may not be possible to transmit periods in a finite message, although many complex numbers do have compact descriptions (*eg* Gaussian rationals, surds). In other cases it may only be possible to transmit an approximation of the periods. If this is so, then Bob knows that his curve is close (in an analytic sense) to the plane curve locus, which is already significant.

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